Aim: Expanding a function in terms of momentum eigenfunction and knowing the "orthonormal" property of the momentum eigenfunction

What is the momentum eigenfunction? It is a non-zero function satisfying $\hat{p} f_p(x) = p f_p(x)$.

We want to show that $f_p(x) = \psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$

Using $p = \hbar k$

$$f_p = \frac{1}{\sqrt{2\pi}} e^{i\frac{px}{\hbar}}$$

$$\hat{p} f_p(x) = \frac{\hbar}{i}\frac{d}{dx}\frac{1}{\sqrt{2\pi}} e^{i\frac{px}{\hbar}}$$

$$= \frac{\hbar}{i}\left(i\frac{p}{\hbar}\right)\left(\frac{1}{\sqrt{2\pi}} e^{\frac{ipx}{\hbar}}\right)$$

$$= p f_p(x).$$

Therefore, $\boxed{\text{the eigenfunctions are } f_p(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} = \psi_k(x)}$

Recall that the Fourier Transform of a function $f(x)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(k) e^{ikx} dk \quad (*)$$

There is only ONE unique $\phi(k)$ in the integral. A function $g(k)$ satisfying $(*)$ inside the integral must be $\phi(k)$.

$\boxed{\text{If we consider the case where } f(x) = e^{ik'x} \text{, we want to find } \phi(k).}$

Let $g(k) = \sqrt{2\pi}\,\delta(k-k')$.

We want to check if $g(k)$ satisfies

$$e^{ik'x} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dk.$$

Checking: $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dk$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(\sqrt{2\pi}\,\delta(k-k')\right) e^{ikx}\, dk$$

$$= \int_{-\infty}^{\infty} \delta(k-k')\, e^{ikx}\, dk$$

$$= e^{ik'x}$$

$$\boxed{\int_{-\infty}^{\infty} \delta(y-y') f(y)\, dy = f(y')}$$

Therefore, $\boxed{\phi(k) = g(k) = \sqrt{2\pi}\,\delta(k-k') \text{ for } f(x) = e^{ik'x}}$ $(**)$

$$\boxed{\begin{array}{l}\text{Then, we want to show} \\ \displaystyle\int_{-\infty}^{\infty} e^{ikx} e^{-ik'x}\, dx = 2\pi\,\delta(k-k').\end{array}}$$

Consider the Fourier Transform of $f(x) = e^{ik'x}$

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx.$$

Since $\phi(k)$ is unique for Fourier Transform,

by $(**)$ $\phi(k) = \sqrt{2\pi}\,\delta(k-k')$.

Therefore, $\sqrt{2\pi}\,\delta(k-k') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik'x} e^{-ikx}\, dx.$

$$2\pi\,\delta(k-k') = \int_{-\infty}^{\infty} e^{ik'x} e^{-ikx}\,dx.$$

Taking complex conjugate,

$$\left[\int_{-\infty}^{\infty} e^{ik'x} e^{-ikx}\,dx\right]^* = 2\pi\,[\delta(k-k')]^*$$

$$\int_{-\infty}^{\infty} \left[e^{ik'x}\right]^* \left[e^{-ikx}\right]^* dx = 2\pi\,\delta(k-k')$$

$$\boxed{\int_{-\infty}^{\infty} e^{-ik'x} e^{ikx}\,dx = 2\pi\,\delta(k'-k)} \qquad \boxed{\begin{aligned}&\delta(k-k')\\ &=\delta(k'-k)\end{aligned}}$$

Then, we have the "orthonormal" property of the momentum eigenfunctions $\psi_k(x)$.

$$\int_{-\infty}^{\infty} \left(\frac{e^{-ik'x}}{\sqrt{2\pi}}\right)\left(\frac{e^{ikx}}{\sqrt{2\pi}}\right) dx = \delta(k'-k).$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \psi_{k'}^*(x)\,\psi_k(x)\,dx = \delta(k'-k).}$$

We can intrepret the Fourier transform of a function as the superposition of different momentum eigenfunctions.

$$\Phi(x) = \underbrace{\int_{-\infty}^{\infty} dx\,\phi(k)}_{\text{Summation of all eigenfunctions' coefficient}}\ \underbrace{\frac{1}{\sqrt{2\pi}}\,e^{ikx}}_{\text{momentum eigenfunction}}$$

This means that every well-behaved function can be expressed as the superposition of momentum eigenfunctions.

SQ18) If a measurement is done, the system stays in a definite state, What we do in real experiment is to prepare huge number of identical copies and do statistic. The more you measure, the better result you get as error goes as $\frac{1}{\sqrt{n}}$.

In this question, we have huge number of identical copies of state:

$$\Psi(x) = \sqrt{\frac{1}{3}}\,\psi_0(x) + \sqrt{\frac{2}{3}}\,\psi_1(x) = c_0\psi_0(x) + c_1\psi_1(x),$$

where $\psi_n(x)$ is the wave function of harmonic oscillator.

Case A: (a) The time evolution of wave function should be:

$$\Psi(x,t) = \sqrt{\frac{1}{3}}\,\psi_0(x)e^{-iE_0t/\hbar} + \sqrt{\frac{2}{3}}\,\psi_1(x)e^{-iE_1t/\hbar}, \quad \text{where } E_n = (n+\tfrac{1}{2})\hbar\omega$$

$$\langle E\rangle = \int_{-\infty}^{\infty} \Psi^*(i\hbar)\frac{\partial}{\partial t}\Psi\,dx$$

$$= \int_{-\infty}^{\infty}\left[\sqrt{\frac{1}{3}}\psi_0^*(x)e^{+iE_0t/\hbar} + \sqrt{\frac{2}{3}}\psi_1^*(x)e^{+iE_1t/\hbar}\right](i\hbar)(-i/\hbar)\left[\sqrt{\frac{1}{3}}\psi_0(x)E_0e^{-iE_0t/\hbar} + \sqrt{\frac{2}{3}}\psi_1(x)E_1e^{-iE_1t/\hbar}\right]dx$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{3}|\psi_0(x)|^2E_0 + \frac{2}{3}|\psi_1(x)|^2E_1\right]dx \qquad * \int_{-\infty}^{\infty}\psi_n(x)\psi_m(x)\,dx = \delta_{nm}$$

$|c_0|^2E_0 + |c_1|^2E_1$

$$= \boxed{\frac{E_0}{3} + \frac{2E_1}{3}}$$

$$= \frac{1}{3}\left(\frac{1}{2}\hbar\omega + 2\cdot\frac{3}{2}\hbar\omega\right)$$

$$= \frac{7}{6}\hbar\omega$$

In principle,

$$\langle E\rangle = \int_{-\infty}^{\infty}\Psi^*\hat{H}_{h.o.}\Psi\,dx$$

$$\langle E^2\rangle = \int_{-\infty}^{\infty}\Psi^*\hat{H}_{h.o.}\hat{H}_{h.o.}\Psi\,dx$$

but it will be very tired to do in this way

* This is the formal way to do it, but by inspection, one can already write $[(\sqrt{\frac{1}{3}})^2E_0 + (\sqrt{\frac{2}{3}})^2E_1]$

(b) $$\langle E^2\rangle = \int_{-\infty}^{\infty}\Psi^*\left[(i\hbar)\frac{\partial}{\partial t}\right]^2\Psi\,dx$$

$$= \int_{-\infty}^{\infty}\left[\sqrt{\frac{1}{3}}\psi_0^*(x)e^{+iE_0t/\hbar} + \sqrt{\frac{2}{3}}\psi_1^*(x)e^{+iE_1t/\hbar}\right](-\hbar^2)\left(\frac{-1}{\hbar^2}\right)\left[\sqrt{\frac{1}{3}}\psi_0(x)E_0^2e^{-iE_0t/\hbar} + \sqrt{\frac{2}{3}}\psi_1(x)E_1^2e^{-iE_1t/\hbar}\right]dx$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{3}|\psi_0(x)|^2E_0^2 + \frac{2}{3}|\psi_1(x)|^2E_1^2\right]dx \qquad * \int_{-\infty}^{\infty}\psi_n(x)\psi_m(x)\,dx = \delta_{nm}$$

$|c_0|^2E_0 + |c_1|^2E_1$

$$= \boxed{\frac{E_0^2}{3} + \frac{2E_1^2}{3}}$$

$$= \frac{1}{3}\left(\frac{1}{4}\hbar^2\omega^2 + 2\cdot\frac{9}{4}\hbar^2\omega^2\right)$$

$$= \frac{19}{12}\hbar^2\omega^2$$

* One can also write $[(\sqrt{\frac{1}{3}})^2E_0^2 + (\sqrt{\frac{2}{3}})^2E_1^2]$

$$\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2}$$

$$= \left[\frac{19}{12}\hbar^2\omega^2 - \frac{49}{36}\hbar^2\omega^2\right]^{\frac{1}{2}}$$

$$= \frac{\sqrt{2}}{3}\hbar\omega$$

In experiment, sometime we will get E_0 and sometime we will get E_1. But NOT $E_2, E_3, E_4, \ldots$ We measure huge number of copies, thus get $\langle E\rangle$. ΔE is the spread.

(b) $\langle x \rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx$ * See appendix, $a = \frac{m\omega}{\hbar}$

$$= \int_{-\infty}^{\infty} \left[\sqrt{\tfrac{1}{3}} \psi_0^*(x) + \sqrt{\tfrac{2}{3}} \psi_1^*(x) \right] x \left[\sqrt{\tfrac{1}{3}} \psi_0(x) + \sqrt{\tfrac{2}{3}} \psi_1(x) \right] dx$$

$$= \int_{-\infty}^{\infty} \left[\frac{\sqrt{2}}{3} \psi_0^*(x) x \psi_1(x) + \frac{\sqrt{2}}{3} \psi_1^*(x) x \psi_0(x) \right] dx$$

$$= \sqrt{\tfrac{2}{3}} \frac{1}{\sqrt{2}} a^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{3}} a^{-\frac{1}{2}}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \, dx$$

$$= \int_{-\infty}^{\infty} \left[\sqrt{\tfrac{1}{3}} \psi_0^*(x) + \sqrt{\tfrac{2}{3}} \psi_1^*(x) \right] x^2 \left[\sqrt{\tfrac{1}{3}} \psi_0(x) + \sqrt{\tfrac{2}{3}} \psi_1(x) \right] dx$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{3} \psi_0^*(x) x^2 \psi_0(x) + \frac{2}{3} \psi_1^*(x) x^2 \psi_1(x) \right] dx$$

$$= \frac{1}{3} \cdot \frac{1}{2} a^{-1} + \frac{2}{3} \cdot \frac{3}{2} a^{-1}$$

$$= \frac{7}{6} a^{-1}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \sqrt{\frac{7}{6} a^{-1} - \frac{1}{3} a^{-1}}$$

$$= \sqrt{\frac{5}{6}} \left(\frac{\hbar}{m\omega} \right)^{\frac{1}{2}}$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^* \left(-i\hbar \frac{d}{dx}\right) \Psi \, dx$$

$$= \int_{-\infty}^{\infty} \left[\sqrt{\tfrac{1}{3}} \psi_0^*(x) + \sqrt{\tfrac{2}{3}} \psi_1^*(x) \right] (-i\hbar) \left[\sqrt{\tfrac{1}{3}} \left(-\sqrt{\tfrac{a}{2}} \psi_1(x)\right) + \sqrt{\tfrac{2}{3}} \left(\sqrt{2a}\, \psi_0(x) - a x \psi_1(x)\right) \right] dx$$

$$= (-i\hbar) \int_{-\infty}^{\infty} \left[\sqrt{\tfrac{1}{3}} \sqrt{\tfrac{2}{3}} \sqrt{2a}\, |\psi_0(x)|^2 - \sqrt{\tfrac{1}{3}} \sqrt{\tfrac{2}{3}}\, a \psi_0^*(x) x \psi_1(x) - \sqrt{\tfrac{2}{3}} \sqrt{\tfrac{1}{3}} \sqrt{\tfrac{a}{2}}\, |\psi_1(x)|^2 \right] dx$$

$$= (-i\hbar) \left[\frac{2}{3} \sqrt{a} - \frac{\sqrt{2}}{3} a \frac{1}{\sqrt{2}} a^{-\frac{1}{2}} - \frac{\sqrt{a}}{3} \right]$$

$$= 0$$

*By inspection, the answer must be 0 as $\langle p \rangle$ cannot be complex.

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \Psi^* \left(-i\hbar \frac{d}{dx}\right)^2 \Psi \, dx$$

$$= -\hbar^2 \int_{-\infty}^{\infty} \left[\sqrt{\tfrac{1}{3}} \psi_0^*(x) + \sqrt{\tfrac{2}{3}} \psi_1^*(x) \right] \left[\sqrt{\tfrac{1}{3}} \left[\frac{a^{\frac{3}{2}}}{\sqrt{2}} x \psi_1(x) - a \psi_0(x) \right] + \sqrt{\tfrac{2}{3}} \left[a^2 x^2 \psi_1(x) - 2a \psi_1(x) - \sqrt{2}\, a^{\frac{3}{2}} x \psi_0(x) \right] \right] dx$$

$$= -\hbar^2 \int_{-\infty}^{\infty} \left[\frac{1}{3} \frac{a^{\frac{3}{2}}}{\sqrt{2}} \psi_0^*(x) x \psi_1(x) - \frac{1}{3} a |\psi_0(x)|^2 + \frac{2}{3} a^2 \psi_1^*(x) x^2 \psi_1(x) - \frac{2}{3} \cdot 2a |\psi_1(x)|^2 - \frac{2\sqrt{2}}{3} a^{\frac{3}{2}} \psi_1^*(x) x \psi_0(x) \right] dx$$

$$= -\hbar^2 \left[\frac{1}{3} \frac{a^{\frac{3}{2}}}{\sqrt{2}} \frac{1}{\sqrt{2}} a^{-\frac{1}{2}} - \frac{1}{3} a + \frac{2}{3} a^2 \frac{3}{2} a^{-1} - \frac{4}{3} a - \frac{2\sqrt{2}}{3} a^{\frac{3}{2}} \frac{1}{\sqrt{2}} a^{-\frac{1}{2}} \right]$$

$$= -\hbar^2 \left[\frac{a}{6} - \frac{a}{3} + a - \frac{4}{3} a - \frac{2}{3} a \right]$$

$$= \frac{7}{6} \hbar^2 a$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$$
$$= \sqrt{\frac{7}{6}}\hbar a^{\frac{1}{2}} = \sqrt{\frac{7}{6}}(m\omega\hbar)^{\frac{1}{2}}$$
$$\Delta x \cdot \Delta p = \sqrt{\frac{5}{6}}\sqrt{\frac{7}{6}}\left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}}(m\omega\hbar)^{\frac{1}{2}}$$
$$= \frac{\sqrt{35}}{6}\hbar \simeq 0.986\hbar \geq \frac{\hbar}{2}$$

Case B: (c) For group 0, all copies are in ψ_0 (Ground state)

It is well known that for ground state 1D harmonic oscillators:
$$\Delta x = \sqrt{\langle x^2\rangle}$$
$$= \sqrt{\frac{\hbar}{2m\omega}}$$
$$\Delta p = \sqrt{\langle p^2\rangle}$$
$$= \sqrt{\frac{m\hbar\omega}{2}}$$
$$\Delta x \Delta p = \frac{\hbar}{2}$$

(d) For group 1, all copies are in ψ_1 (1st excited state)
$$\langle x\rangle = \int_{-\infty}^{\infty} \psi_1^* \, x \, \psi_1 \, dx$$
$$= 0$$
$$\langle x^2\rangle = \int_{-\infty}^{\infty} \psi_1^* x^2 \psi_1 \, dx$$
$$= \frac{3}{2}a^{-1}$$
$$\langle p\rangle = \int_{-\infty}^{\infty} \psi_1^* \left(-i\hbar\frac{d}{dx}\right)\psi_1 \, dx$$
$$= (-i\hbar)\int_{-\infty}^{\infty} \psi_1^* \left(\sqrt{2a}\,\psi_0(x) - ax\,\psi_1(x)\right)dx$$
$$= 0$$

* $\langle p\rangle$ cannot be complex thus it must be 0.
$$\langle p^2\rangle = \int_{-\infty}^{\infty} \psi_1^* \left(-i\hbar\frac{d}{dx}\right)^2 \psi_1 \, dx$$
$$= -\hbar^2\int_{-\infty}^{\infty} \psi_1^* \left(a^2x^2\psi_1(x) - 2a\psi_1(x) - \sqrt{2}a^{\frac{3}{2}}x\,\psi_0(x)\right)dx$$
$$= -\hbar^2\left[a^2\,\frac{3}{2}a^{-1} - 2a - \sqrt{2}a^{\frac{3}{2}}\,\frac{1}{\sqrt{2}}a^{-\frac{1}{2}}\right]$$
$$= -\hbar^2\left[\frac{3}{2}a - 2a - a\right]$$
$$= \frac{3}{2}\hbar^2 a$$
$$\Delta x = \sqrt{\langle x^2\rangle} = \sqrt{\frac{3}{2}}a^{-\frac{1}{2}}$$
$$\Delta p = \sqrt{\langle p^2\rangle} = \sqrt{\frac{3}{2}}\hbar a^{\frac{1}{2}}$$

$\Delta x \Delta p = \frac{3}{2}\hbar > \frac{1}{2}\hbar$

$\boxed{\Delta x \Delta p = \frac{\hbar}{2}(2n+1)}$ for different states

(e) ① Do the average of group 0 and 1:

$(\Delta x \Delta p)_1 = (\frac{3}{2}\hbar + \frac{1}{2}\hbar)/2$

$= \hbar \approx 0.986\hbar$ (Case A)

② Doing weighting:

$(\Delta x \Delta p)_3 = \frac{1}{3}\cdot\frac{1}{2}\hbar + \frac{2}{3}\cdot\frac{3}{2}\hbar$

$\approx 1.17\hbar > (\Delta x \Delta p)_1$

Both methods have a larger $\Delta x \Delta p$, which satisfy $\Delta x \Delta p \geq \frac{\hbar}{2}$.

However, both cannot get the correct value of the expected uncertainty $\Delta x \Delta p \approx 0.986\hbar$

(f) In (c) and (d), the value of $\Delta x \Delta p$ is only for the corresponding eigen state. (c) is for ground state and (d) is for 1st excited state. Thus they are different. The uncertainty of different energy state is $\Delta x \Delta p = \frac{\hbar}{2}(2n+1)$.
Nonetheless for (b), the system is in superposition of ground state ψ_0 and 1st excited state ψ_1. We should also consider the cross term ψ_0 and ψ_1.
So (a) is different from (b), (c).

	(a)	(b)	(c)
$\Delta x \Delta p$ of:	$\Psi(x) = c_0\psi_0(x) + c_1\psi_1(x)$	$\psi_0(x)$	$\psi_1(x)$

$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$ $\qquad$ $\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-\frac{m\omega x^2}{2\hbar}}$ $\qquad$ $a = \frac{m\omega}{\hbar}$

e^{-ax^2} is a even function, $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$

xe^{-ax^2} is a odd function, $\int_{-\infty}^{\infty} x e^{-ax^2} dx = 0$

$x^2e^{-ax^2}$ is a even function, $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = -\frac{d}{da}\int_{-\infty}^{\infty} e^{-ax^2} dx = -\frac{d}{da}\sqrt{\frac{\pi}{a}} = \frac{\sqrt{\pi}}{2} a^{-\frac{3}{2}}$

$x^3e^{-ax^2}$ is a odd function, $\int_{-\infty}^{\infty} x^3 e^{-ax^2} dx = 0$

$x^4e^{-ax^2}$ is a even function, $\int_{-\infty}^{\infty} x^4 e^{-ax^2} dx = \frac{d^2}{da^2}\int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{d^2}{da^2}\sqrt{\frac{\pi}{a}} = \frac{3}{4}\sqrt{\pi}\, a^{-\frac{5}{2}}$

$x^5e^{-ax^2}$ is a odd function, $\int_{-\infty}^{\infty} x^5 e^{-ax^2} dx = 0$

$x^6e^{-ax^2}$ is a even function, $\int_{-\infty}^{\infty} x^6 e^{-ax^2} dx = -\frac{d^3}{da^3}\int_{-\infty}^{\infty} e^{-ax^2} dx = -\frac{d^3}{da^3}\sqrt{\frac{\pi}{a}} = \frac{15}{8}\sqrt{\pi}\, a^{-\frac{7}{2}}$ ……

$\therefore \int_{-\infty}^{\infty} \psi_0^*(x)\, x\, \psi_1(x)\, dx = a^{\frac{1}{2}} \sqrt{2a} \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx \cdot \pi^{-\frac{1}{2}}$

$= \sqrt{2}\, a\, \frac{1}{2} a^{-\frac{3}{2}} = \frac{1}{\sqrt{2}} a^{-\frac{1}{2}} = \int_{-\infty}^{\infty} \psi_1^*(x)\, x\, \psi_0(x)\, dx$

$\int_{-\infty}^{\infty} \psi_0^*(x)\, x^2\, \psi_0(x)\, dx = a^{\frac{1}{2}} \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx \cdot \pi^{-\frac{1}{2}}$

$= a^{\frac{1}{2}} \frac{1}{2} a^{-\frac{3}{2}} = \frac{1}{2} a^{-1}$

$\int_{-\infty}^{\infty} \psi_1^*(x)\, x^2\, \psi_1(x)\, dx = a^{\frac{1}{2}} \cdot 2a \int_{-\infty}^{\infty} x^4 e^{-ax^2} dx \cdot \pi^{-\frac{1}{2}}$

$= 2a^{\frac{3}{2}} \frac{3}{4} a^{-\frac{5}{2}}$

$= \frac{3}{2} a^{-1}$

$\frac{d}{dx}\psi_0(x) = \left(\frac{a}{\pi}\right)^{\frac{1}{4}} (-ax) e^{-\frac{a}{2}x^2} = -\sqrt{\frac{a}{2}}\, \psi_1(x)$

$\frac{d^2}{dx^2}\psi_0(x) = -\sqrt{\frac{a}{2}} \frac{d}{dx}\psi_1(x)$

$= \frac{a^{\frac{3}{2}}}{\sqrt{2}} x\psi_1(x) - a\psi_0(x)$

$\frac{d}{dx}\psi_1(x) = \left(\frac{a}{\pi}\right)^{\frac{1}{4}} \sqrt{2a} \left[e^{-\frac{a}{2}x^2} - ax^2 e^{-\frac{a}{2}x^2}\right]$

$= \sqrt{2a}\,\psi_0(x) - ax\psi_1(x)$

$\frac{d^2}{dx}\psi_1(x) = \sqrt{2a}\frac{d}{dx}\psi_0(x) - a\frac{d}{dx}[x\psi_1(x)]$

$= -a\psi_1(x) - ax(\sqrt{2a}\psi_0(x) - ax\psi_1(x)) - a\psi_1(x)$

$= a^2x^2\psi_1(x) - 2a\psi_1(x) - \sqrt{2}a^{\frac{3}{2}}x\psi_0(x)$

$\therefore \int_{-\infty}^{\infty} \psi_0^*(x)\psi_1(x)\, dx = 0$

$\int_{-\infty}^{\infty} |\psi_0(x)|^2 dx = \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-ax^2} dx$

$= 1$

$\int_{-\infty}^{\infty} |\psi_1(x)|^2 dx = \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \sqrt{2a} \int_{-\infty}^{\infty} x e^{-ax^2} dx = 1$ by normalization condition

Appendix